Critical Portraits of Complex Polynomials

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Coworkers

PhD Dissertation (2015): *On the Simplest Lamination of a Given Identity Return Triangle*
Brandon L. Barry

UG Posters: (2017) *Critical Portraits and Weakly BiColored Trees*
(2018) *Ambiguous or Non-Generic Critical Portraits of Complex Polynomials*
David J. George and Simon D. Harris

MS Thesis (2017): *Exponential Laminations*

Patrick B. Hartley

Work in Progress

Outline

Outline

Outline

Complex Polynomials

- Polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$:

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

- Compactify $\mathbb{C}$ to $\mathbb{C}_\infty$.
- For P, ∞ is attracting fixed point: for $z \in \mathbb{C}$ with $|z|$ sufficiently large,

$$\lim_{n\to\infty} P^n(z) = \infty.$$

- Basin of attraction of ∞:

$$B_\infty := \{z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) = \infty\}$$

- B_∞ is an open set.

Julia and Fatou Sets

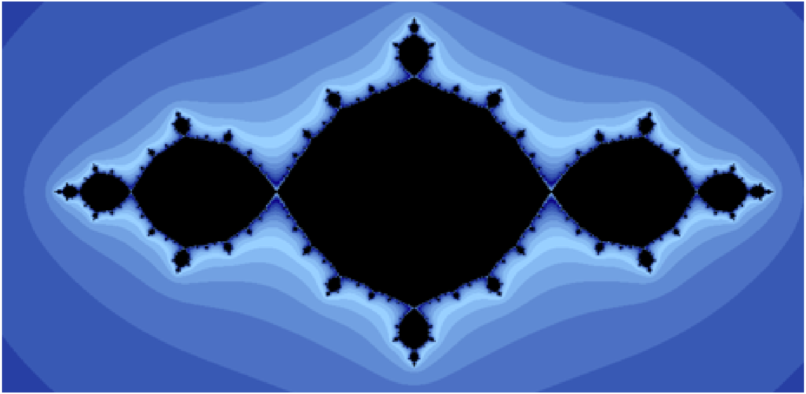
Definitions:

- Julia set $J(P) :=$ boundary of B_∞.
- Fatou set $F(P) := \mathbb{C}_\infty \setminus J(P)$.
- Filled Julia set $K(P) := \mathbb{C}_\infty \setminus B_\infty$.

Fun Facts:

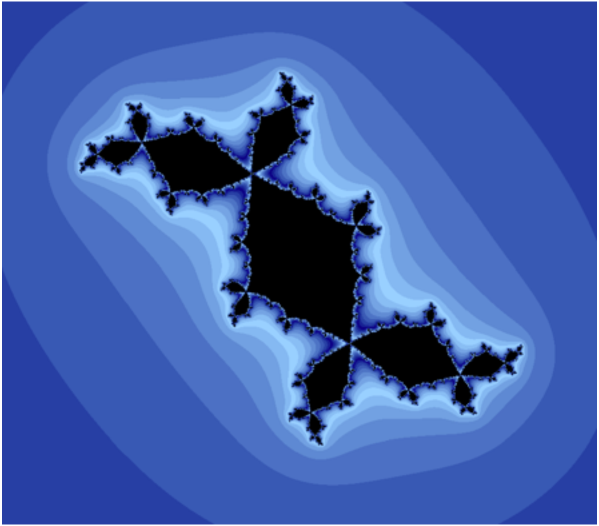
- $J(P)$ is nonempty, compact, and perfect.
- $K(P)$ does not separate $\mathbb{C}$.
- Attracting orbits are in Fatou set.
- Repelling orbits are in Julia set.
- We will assume $J(P)$ is connected (a continuum: compact, connected metric space).

Basillica $z \mapsto z^2 - 1$

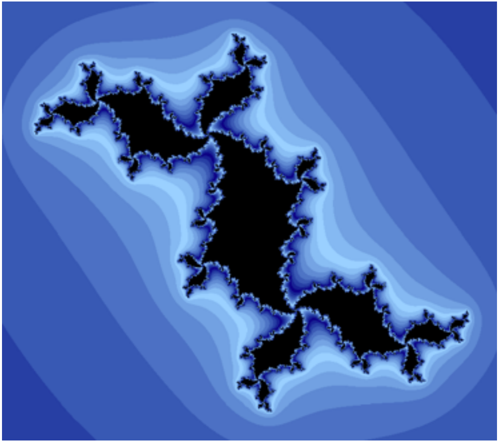


Julia set pictures by Fractalstream

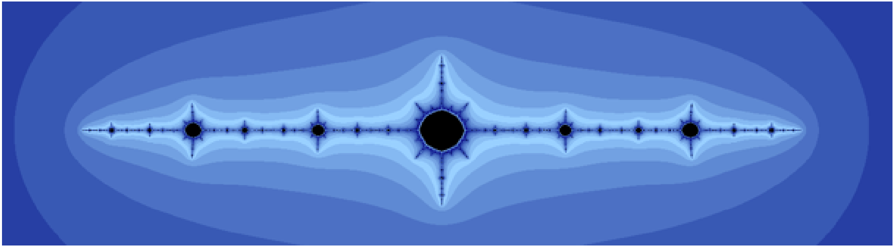
Douady Rabbit $z \mapsto z^2 + (-0.12 + 0.78i)$



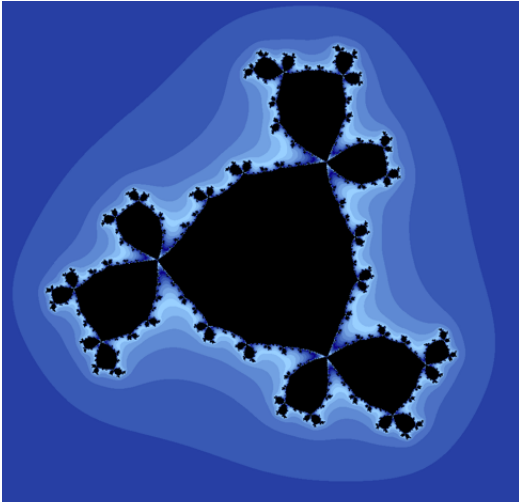
Twisted Rabbit $z \mapsto z^2 + (0.057 + 0.713i)$



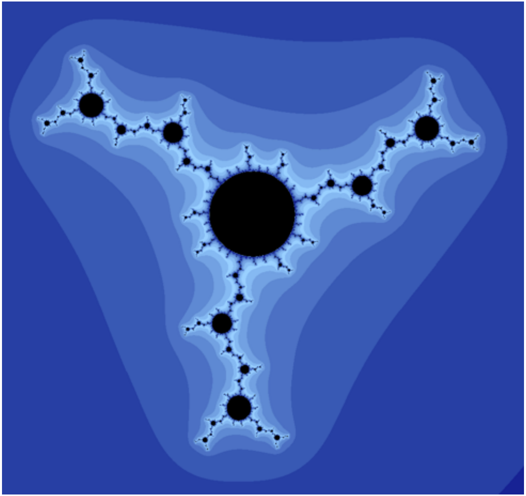
Airplane $z \mapsto z^2 - 1.75$



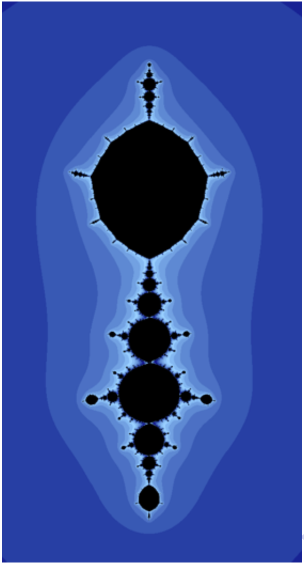
Minnie Mouse $z \mapsto z^3 + (0.545 + 0.539i)$



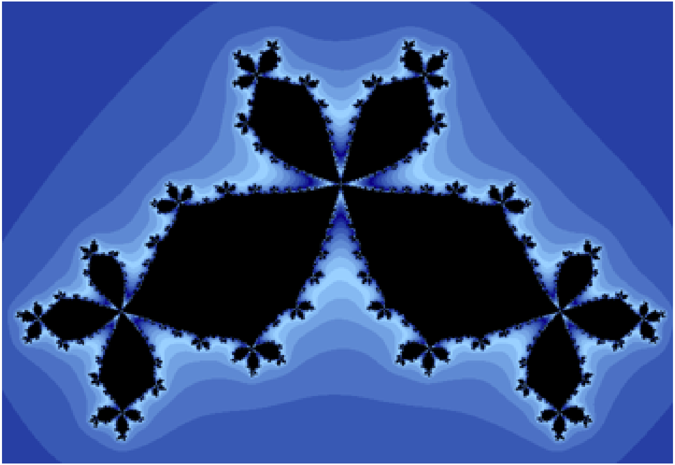
Helicopter $z \mapsto z^3 + (-0.2634 - 1.2594i)$



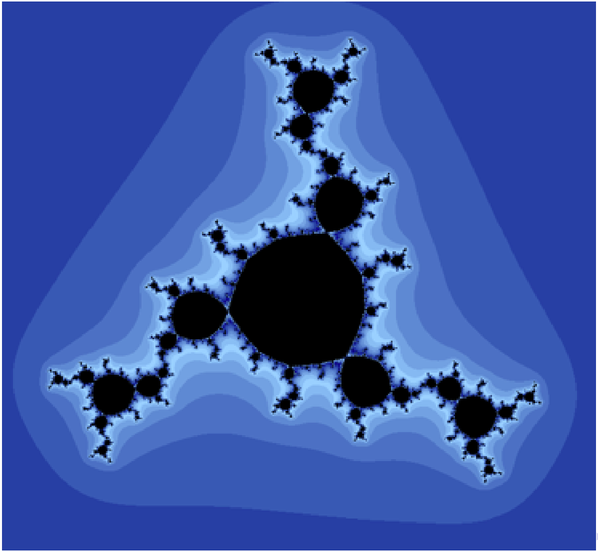
Scorpion/Scepter $z \mapsto z^3 + 3(0.785415i)z^2$



Butterfly $z \mapsto z^3 + 3(-0.5)z^2 + (0.75 + 0.661438i)$

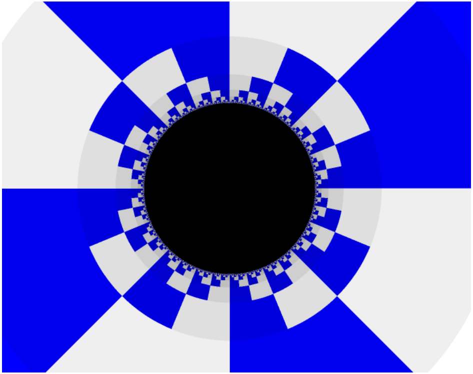


Ninja Throwing Star $z \mapsto z^3 + (0.20257 + 1.095i)$



The Simplest Julia Set – the Unit Circle $\partial\mathbb{D}$

$P(z) = z^2$ $\qquad re^{2\pi i\, t} \mapsto r^2 e^{2\pi i\, 2t}$



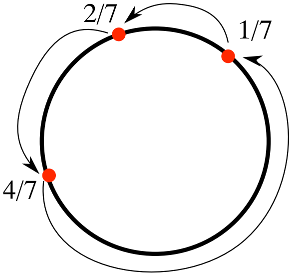
The complement $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ of the closed unit disk is the basin of attraction, B_∞, of infinity.

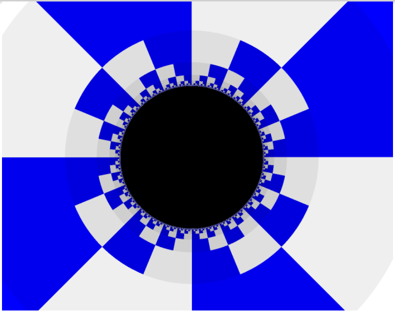
Dynamics on the Unit Circle

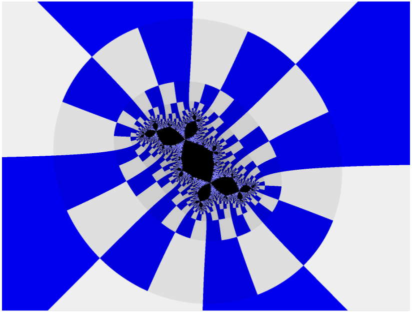
- Consider $P(z) = z^d$ on the unit circle $\partial\mathbb{D}$.
- $z = re^{2\pi t} \mapsto r^d e^{2\pi i(dt)} \longrightarrow$ Angle $2\pi t \mapsto 2\pi(dt)$.
- Measure angles in revolutions:
  Points on $\partial\mathbb{D}$ are coordinatized by $[0, 1)$.

$$\sigma_d : t \mapsto dt \quad (\text{mod } 1) \text{ on } \partial\mathbb{D}$$

Example $d = 2$:



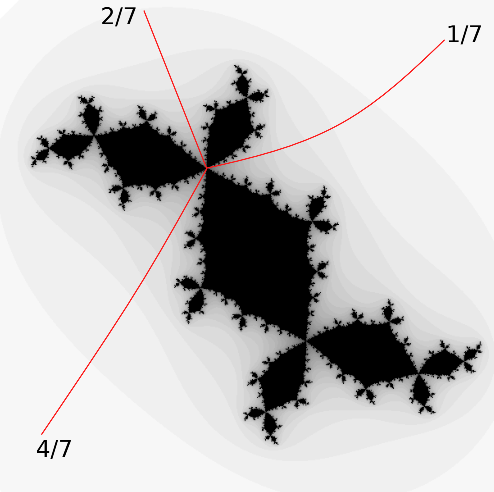




Bottcher's Theorem

$$\begin{array}{ccc} \mathbb{D}_\infty & \xrightarrow{z \mapsto z^d} & \mathbb{D}_\infty \\ \downarrow \phi & & \downarrow \phi \\ B_\infty & \xrightarrow[P]{} & B_\infty \end{array}$$

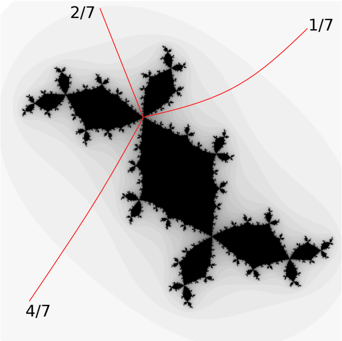
External Rays $P(z) = z^2 + (-0.12 + 0.78i)$



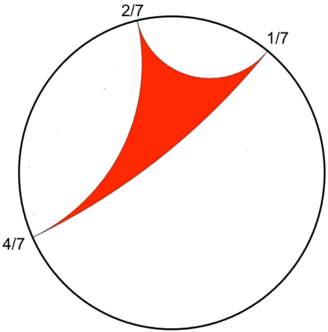
External Rays $\longrightarrow$ Laminations

- Laminations were introduced by William Thurston as a way of encoding connected polynomial Julia sets.

Coincident external rays

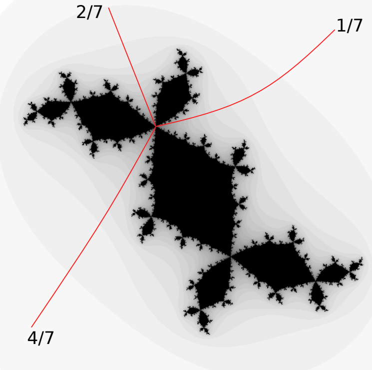


Rabbit triangle



The Rabbit Lamination

The rabbit Julia set



The rabbit lamination

Hyperbolic lamination pictures courtesy of Logan Hoehn

Laminations of the Unit Disk $\mathbb{D}$

Definition

- A *lamination* $\mathcal{L}$ is a collections of chords of $\overline{\mathbb{D}}$, which we call *leaves*, with the property that any two leaves meet, if at all, in a point of $\partial\mathbb{D}$, and
- such that $\mathcal{L}$ has the property that

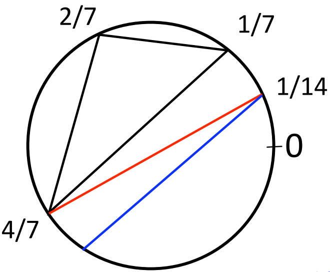
$$\mathcal{L}^* := \partial\mathbb{D} \cup \{\cup\mathcal{L}\}$$

  is a closed subset of $\overline{\mathbb{D}}$.
- We allow *degenerate* leaves – points of $\partial\mathbb{D}$.

Note that $\mathcal{L}^*$ is a continuum: compact, connected metric space.

Extending σ_d to Leaves

- If $\ell \in \mathcal{L}$ is a leaf, we write $\ell = \overline{ab}$, where a and b are the endpoints of ℓ in $\partial\mathbb{D}$.
- We define $\sigma_d(\ell)$ to be the chord $\overline{\sigma_d(a)\sigma_d(b)}$.
- The *length* of a chord is the length of the shorter arc of the circle subtended.
- If it happens that $\sigma_d(a) = \sigma_d(b)$, then $\sigma_d(\ell)$ is a point, called a *critical value* of $\mathcal{L}$, and we say ℓ is a *critical* leaf.



Making the Lamination dynamic!

Definition (Sibling Invariant Lamination)

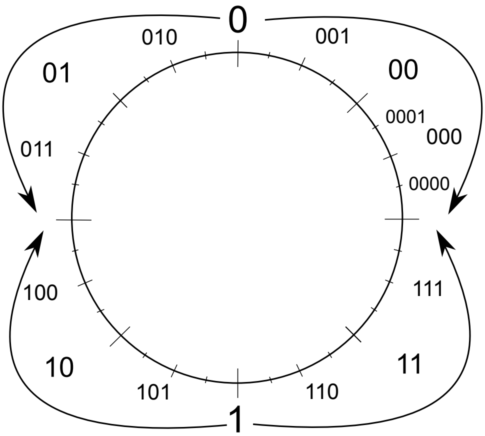
A lamination $\mathcal{L}$ is said to be *sibling d-invariant* provided that:

1. (Forward Invariant) For every $\ell \in \mathcal{L}$, $\sigma_d(\ell) \in \mathcal{L}$.
2. (Backward Invariant) For every non-degenerate $\ell' \in \mathcal{L}$, there is a leaf $\ell \in \mathcal{L}$ such that $\sigma_d(\ell) = \ell'$.
3. (Sibling Invariant) For every $\ell_1 \in \mathcal{L}$ with $\sigma_d(\ell_1) = \ell'$, a non-degenerate leaf, there is a *full sibling collection* of pairwise disjoint leaves $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset \mathcal{L}$ such that $\sigma_d(\ell_i) = \ell'$.

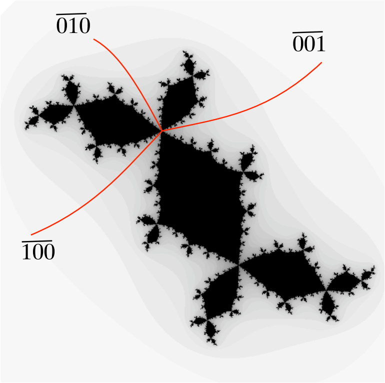
Conditions (1), (2) and (3) allow generating a sibling invariant lamination from a finite amount of initial data.

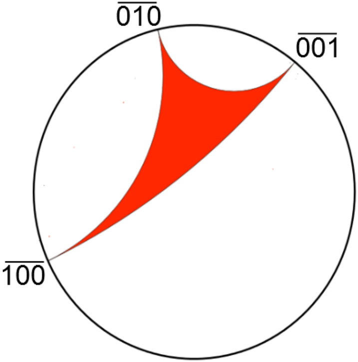
σ_2 Binary Coordinates

Location dynamically defined.



σ_2 Binary Coordinates and Rabbit



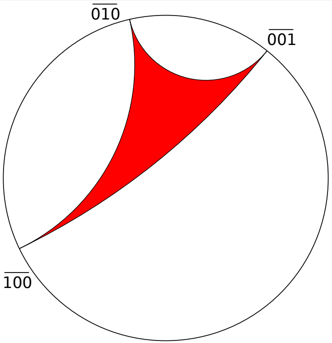


In binary coordinates, σ_2 is the "forgetful" shift.
The overline means the coordinates repeat.

Generating a Lamination from Finite Initial Data

Definition (Pullback Scheme)

A *pullback scheme* for σ_d is a collection of d branches $\tau_1, \tau_2, \ldots, \tau_d$ of the inverse of σ_d whose ranges partition $\partial\mathbb{D}$.

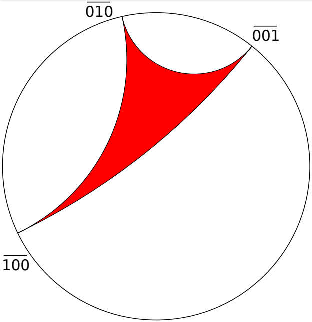


Data: Forward invariant lamination.

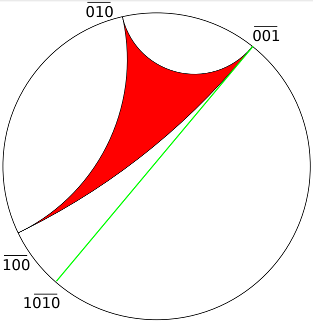
Pullback Scheme

Definition (Guiding Critical Chords)

The generating data of a pullback scheme are a *forward invariant periodic collection of leaves* and a collection of *d* interior disjoint *guiding critical chords*.

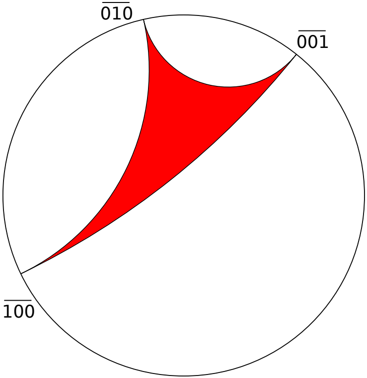


Data: Forward invariant lamination.

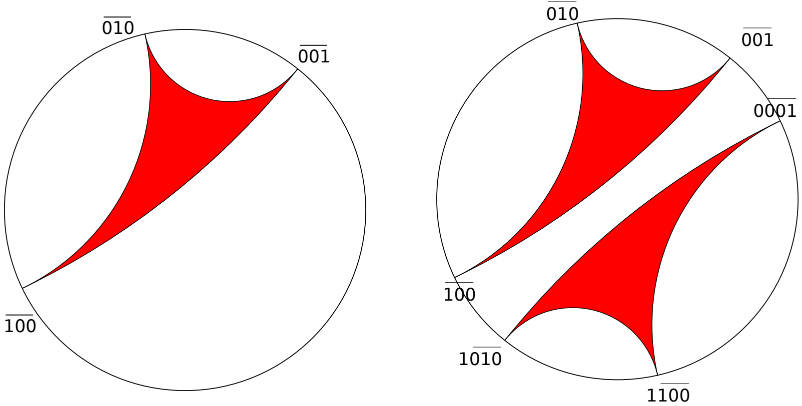


Guiding critical chord(s).

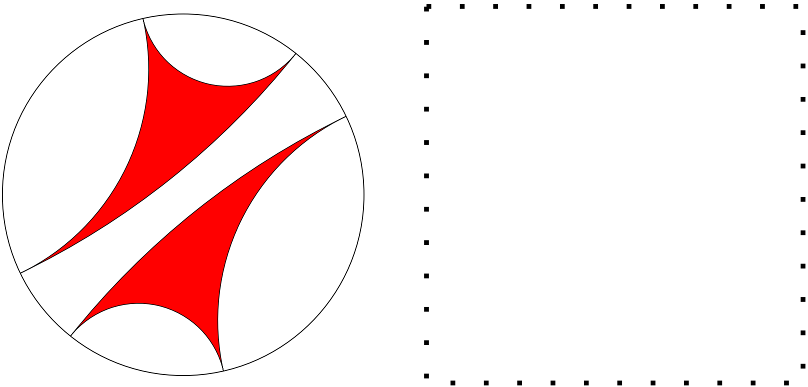
Pullback Sequence



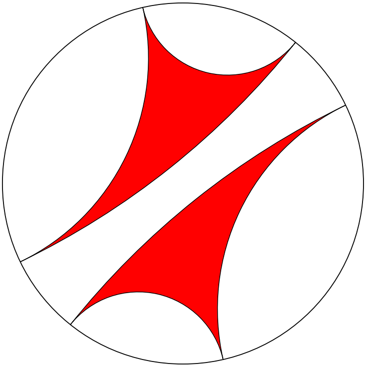
Pullback Sequence

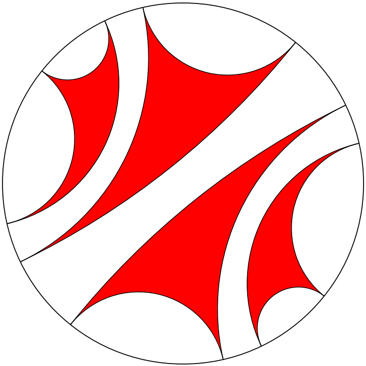


Pullback Sequence

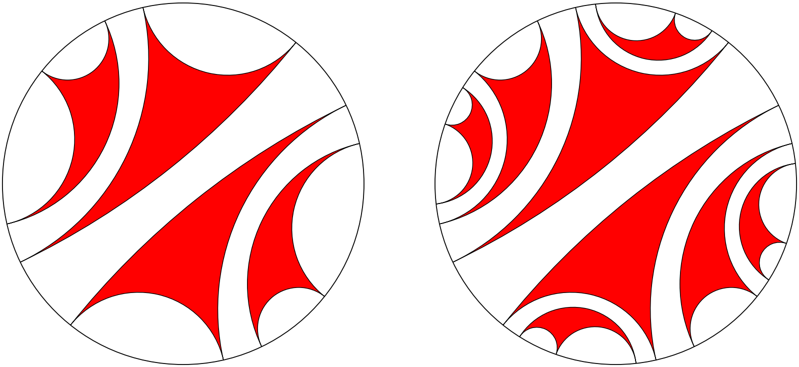


Pullback Sequence

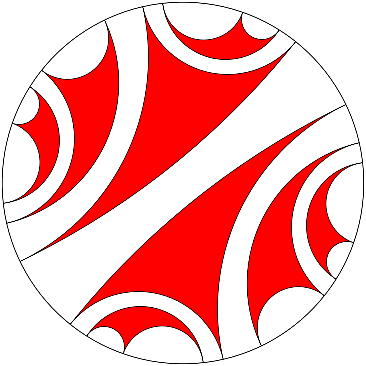


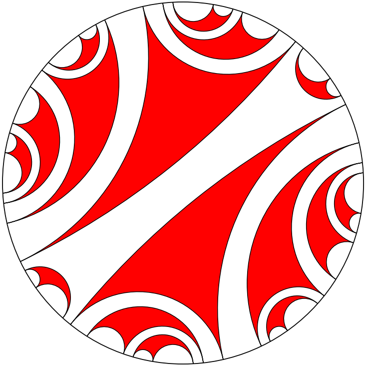


Pullback Sequence

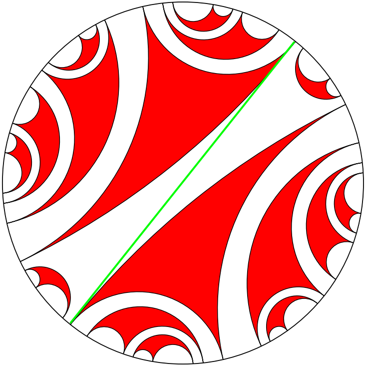


Pullback Sequence

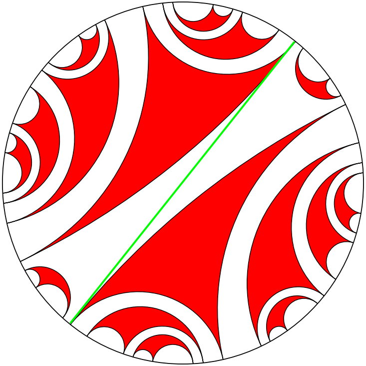


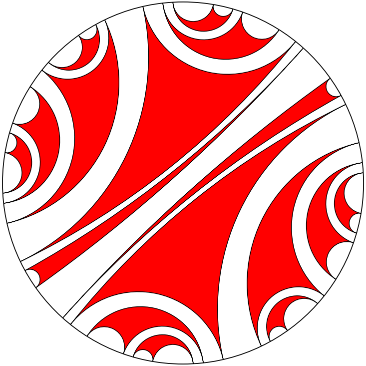


Importance of Guiding Crital Chord



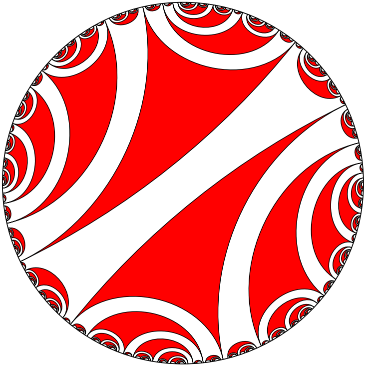
Ambiguity





Quadratic Lamination and Julia Set

Rabbit Lamination



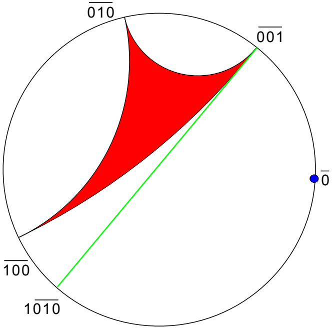
Rabbit Julia Set

Quotient space in plane $\Longrightarrow$ homeomorphic to rabbit Julia set.

Semiconjugate dynamics

Lamination Data for Rabbit Lamination

The critical chord and one endpoint determine the lamination.

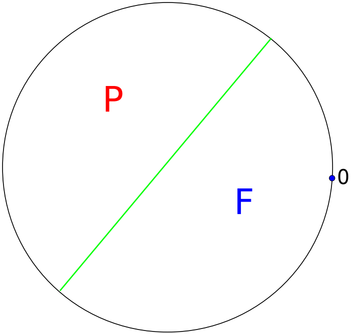


- The rabbit triangle's vertices are the only periodic orbit that stays in the left half.
- The fixed point $\overline{0}$ is the only periodic orbit that stays in the right half.

Critical Portrait $\longrightarrow$ Dual Graph

Abstract from the Lamination Data just the critical chord.

Bicolored Critical Portrait



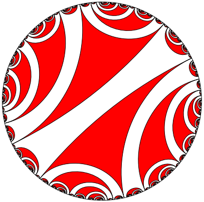
Bicolored Dual Graph

Cast of Characters

Julia Set

→

Lamination



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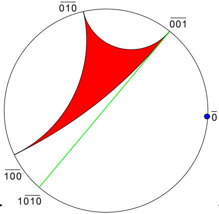
Lamination Data



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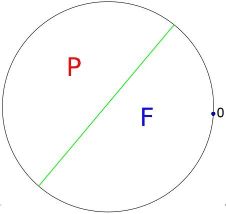


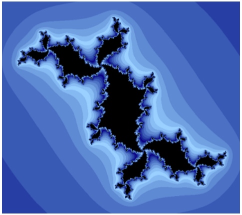
Critical Portrait

→

Bicolored Tree

Cast of Characters

Julia Set



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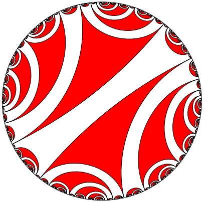
Lamination



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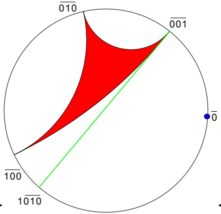
Lamination Data



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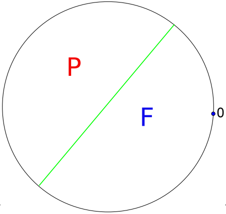

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Critical Portrait

Bicolored Tree

The Simplest Lamination

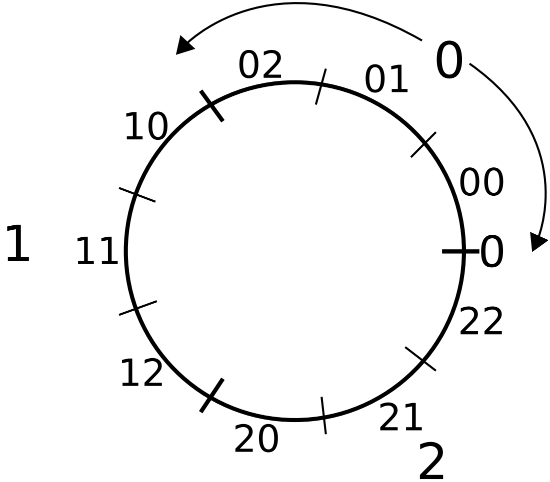
Definition

For given lamination data for σ_d consisting of a collection of periodic polygons and guiding critical chords, we call a pull-back lamination whose Fatou domains
(1) are bordered by sides of the given polygons, and
(2) contain the guiding critical chords,
a *simplest lamination* for the given data.

- There is no claim that a simplest lamination is unique, though that would be a desireable consequence of a good definition.
- See Brandon Barry's dissertation:
  Theorem. For σ_3, there is always a simplest lamination.

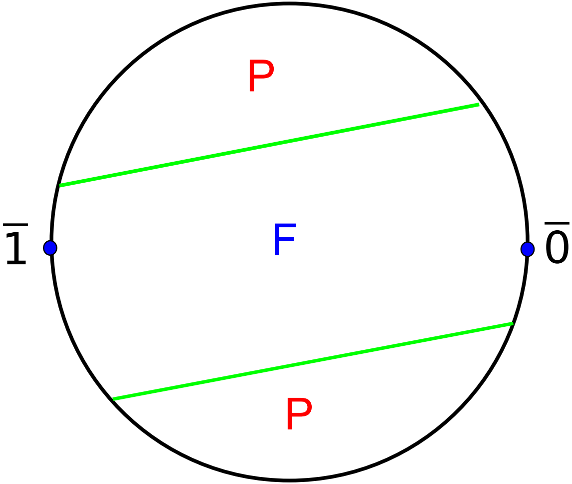
σ_3 and Ternary Coordinates

Ternary coordinates correspond to shift σ_3.



Critical Chords, Critical Sectors, and Fixed Points

Example for σ_3 (angle-tripling):



Weakly Bicolored Trees

Theorem: Critical portraits correspond dually to weakly bicolored trees. [George, Harris]

Definition

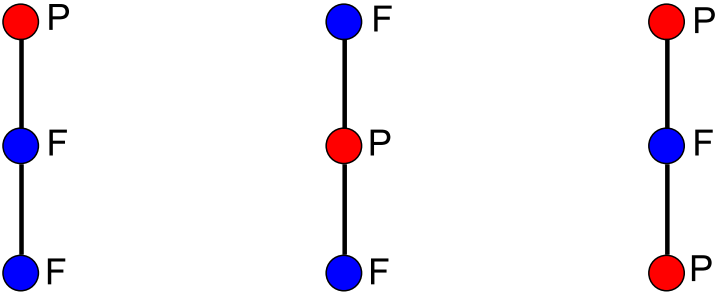
A tree is said to be *weakly bicolored* provided it satisfies the following conditions:

1. Each of two vertex colors (say, red and blue) is used at least once.
2. One vertex color, say blue, can be adjacent to itself.
3. One vertex color, say red, cannot be adjacent to itself.

Problem: How many different weakly bicolored trees are there, up to orientation-preserving planar isomorphism, with n vertices?

Weakly Bicolored Trees

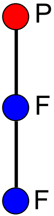
Below are the three possible weakly bicolored trees on three vertices up to orientation-preserving planar isomorphism:



- Graphs corresponding dually to critical portraits are always trees.
- Critical portraits that produce equivalent laminations are rotations of each other.

Role Reversal: P-F-F

Bicolored Tree



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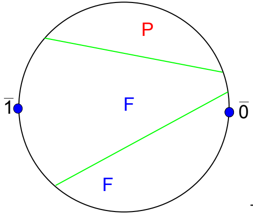
Critical Portrait



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Lamination Data →

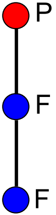
→ →

Lamination

Julia Set

Role Reversal: P-F-F

Bicolored Tree



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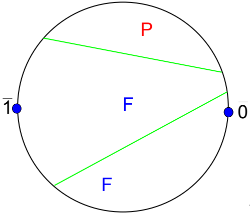
Critical Portrait



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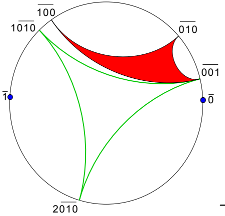
Lamination Data



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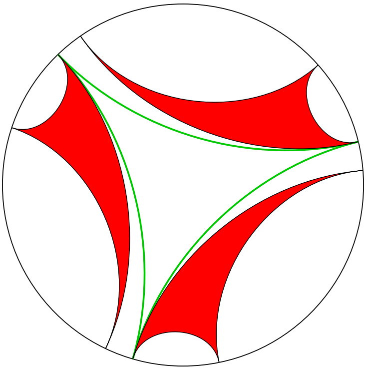
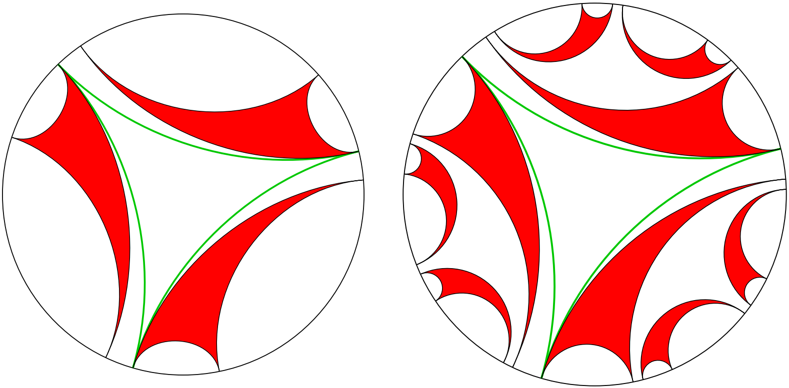
Lamination

Julia Set

First two Pullbacks of Triangle

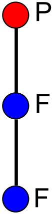


First two Pullbacks of Triangle



Role Reversal: P-F-F

Bicolored Tree



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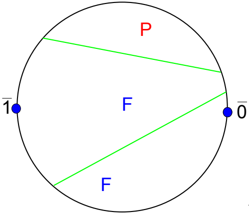
Critical Portrait



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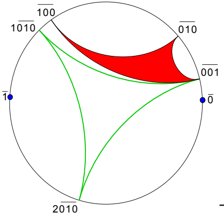
Lamination Data



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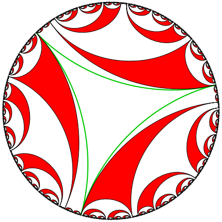

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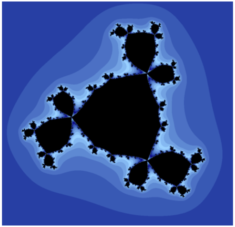
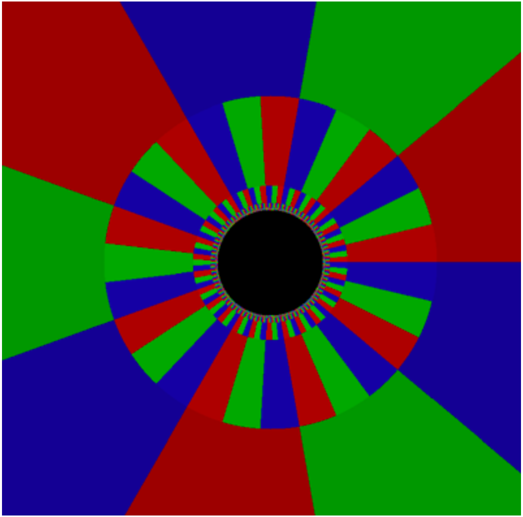


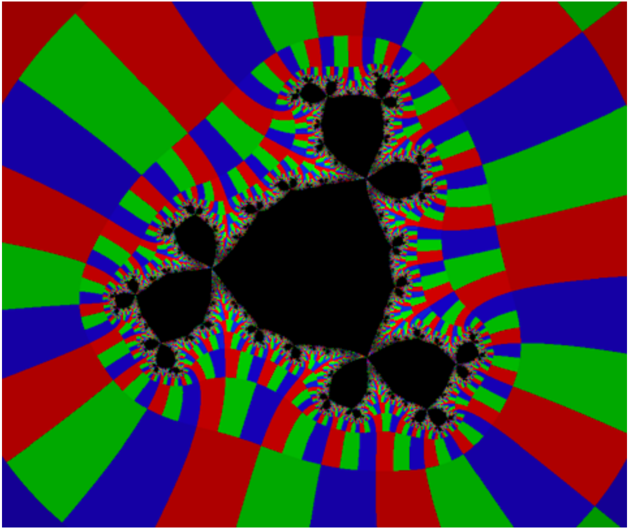
Lamination

Julia Set

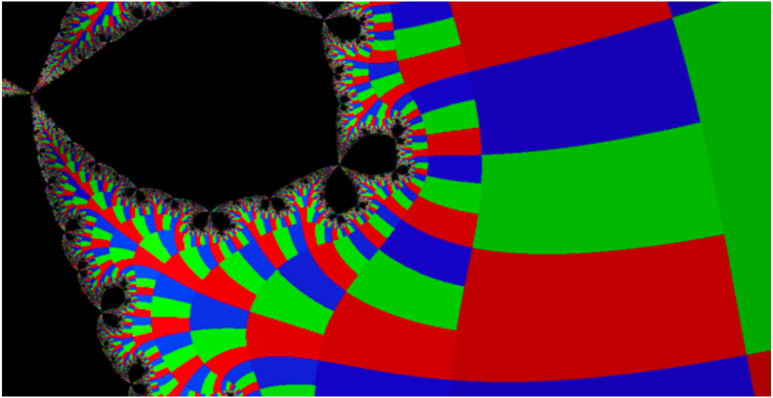
Ternary Checkerboard



Verifying Angles

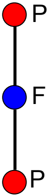


Verifying Angles

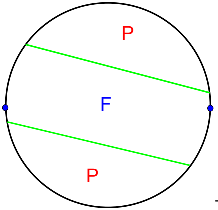


Role Reversal: P-F-P

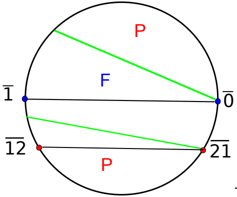
Bicolored Tree



Critical Portrait



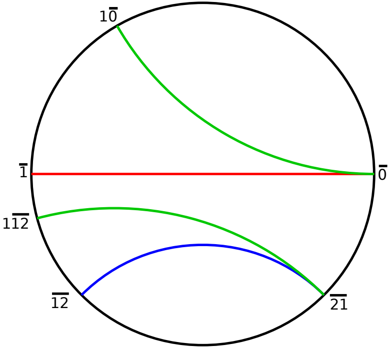
Lamination Data



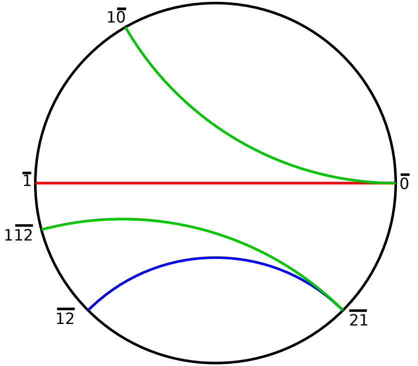
Lamination

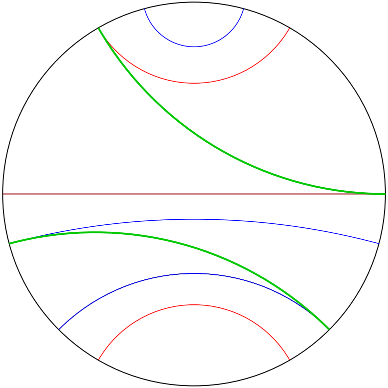
Julia Set

Pullback Sequence

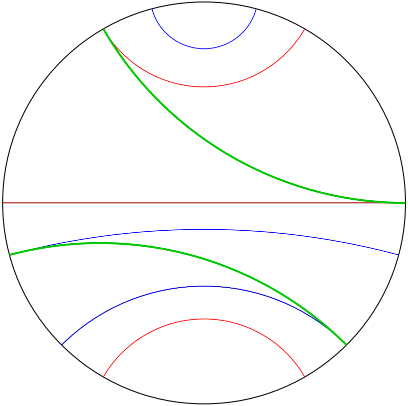


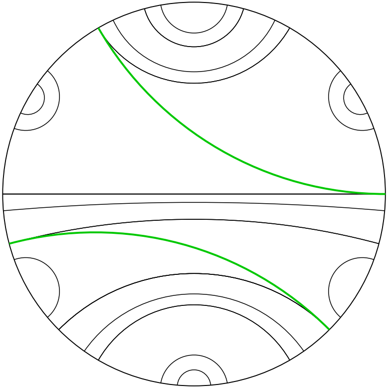
Pullback Sequence





Pullback Sequence





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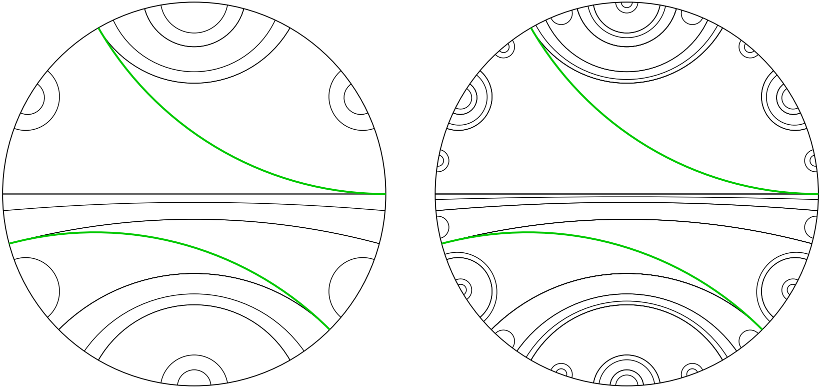
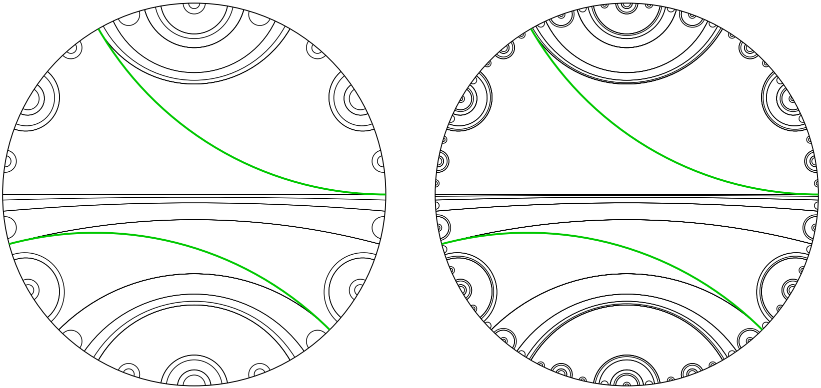


Pullback Sequence



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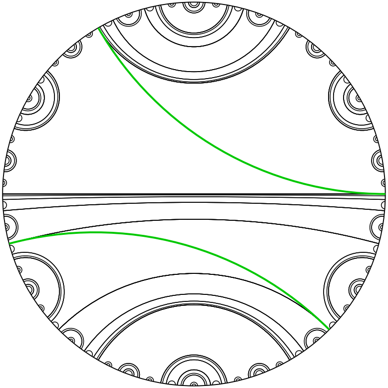
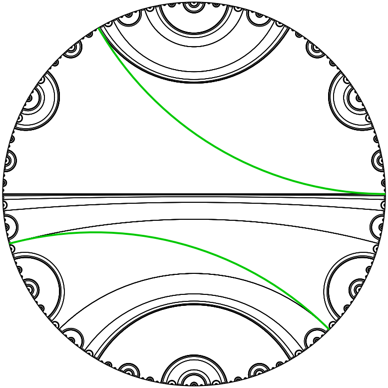
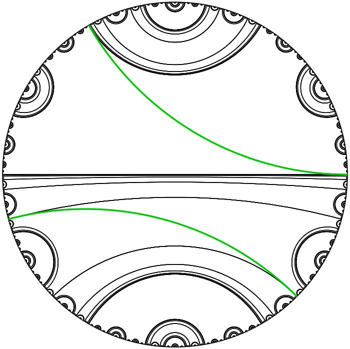


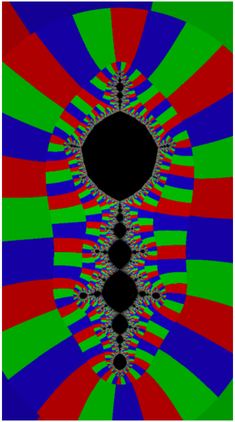


Pullback Sequence

Scorpion Lamination

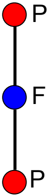


Scorpion Julia set



Role Reversal: P-F-P

Bicolored Tree



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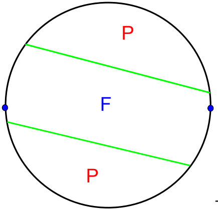
Critical Portrait



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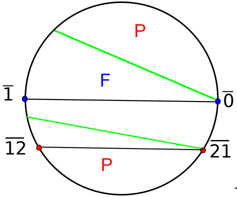
Lamination Data



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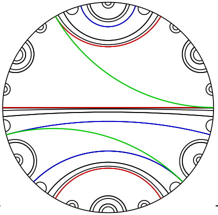

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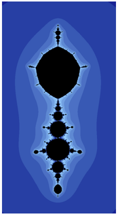
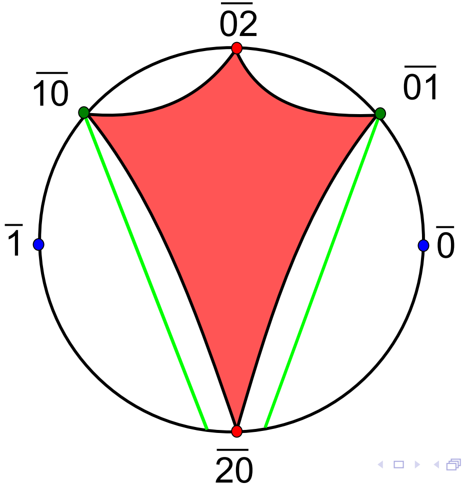
Lamination



Julia Set

Finding the Diamond

Can we find a cubic polynomial and a resulting Julia set incorporating the diamond?



Finding the Diamond

- The lamination data is enough to find the diamond Julia set among a parameterized family of cubic polynomials:
  $z \mapsto z^3 + 3az^2 + b$, for $(a, b) \in \mathbb{C}^2$.
- Two period 2 Fatou domains $\longrightarrow$
- Two period 2 critical points $\longrightarrow$ 0 and $-2a$ $\longrightarrow$
- Two simultaneous equations in parameters a and b $\longrightarrow$
- Multiple specific parameters (a, b):

set a to -0.5. set b to 0.75 + 0.661438i.
set a to -0.5. set b to 0.75 - 0.661438i.
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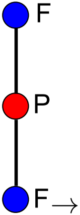
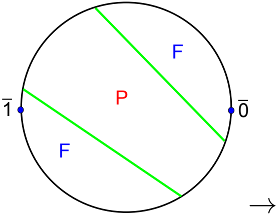
Role Reversal: F-P-F (Diamond)

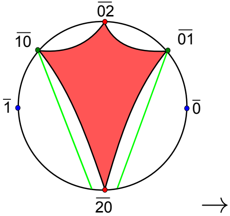
Bicolored Tree



Critical Portrait



Lamination Data

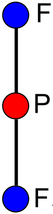


Lamination

Julia Set

Role Reversal: F-P-F (Diamond)

Bicolored Tree



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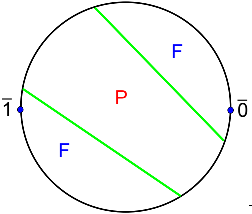
Critical Portrait



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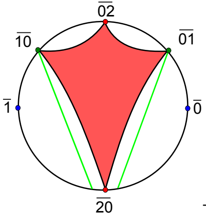
Lamination Data



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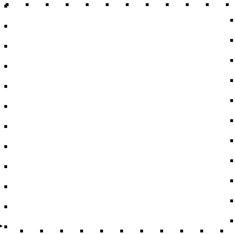
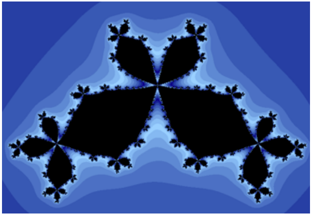
→  →

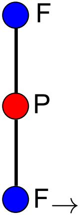


Lamination

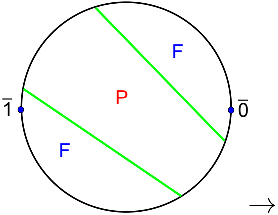
Julia Set (Butterfly)

Role Reversal: F-P-F (Diamond)

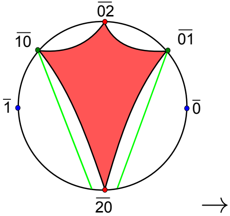
Bicolored Tree

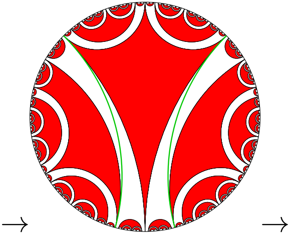


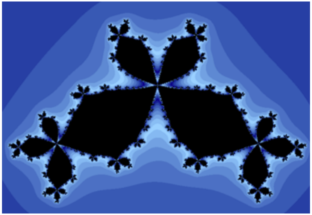
Critical Portrait



Lamination Data



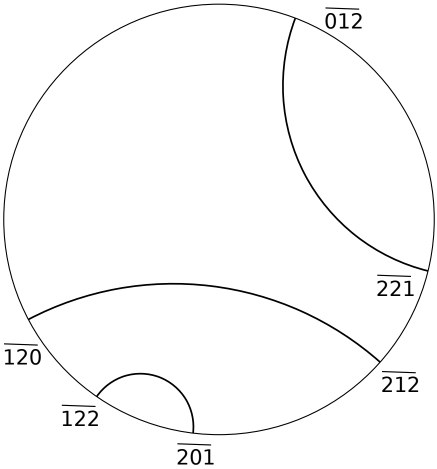




Lamination

Julia Set (Butterfly)

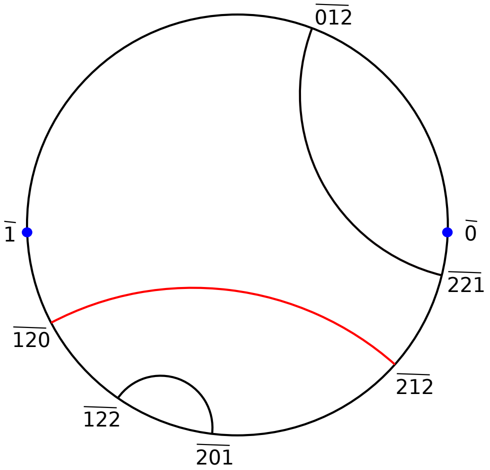
Periodic Lamination Data



What about guiding critical chords?

cf: Brandon Barry

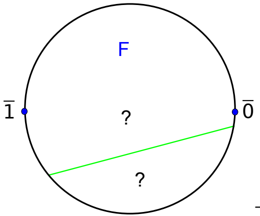
Periodic Lamination Data



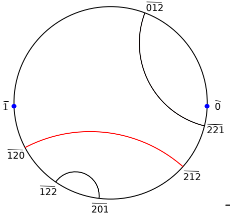
Consider fixed points and chord closest to critical length.

Role Reversal: ?-?-?

Bicolored Tree → Critical Portrait



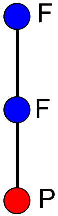
→ Periodic Lamination Data



→ Lamination → Julia Set

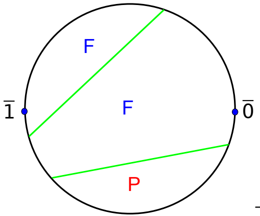
Role Reversal: F-F-? or F-?-F

Bicolored Tree



→

Critical Portrait



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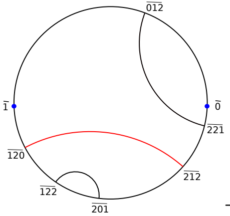
Periodic Lamination Data



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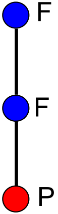
Lamination

Julia Set

Role Reversal: F-F-P

Bicolored Tree



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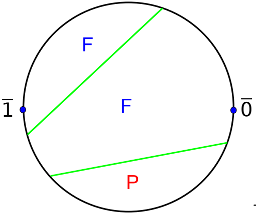
Critical Portrait



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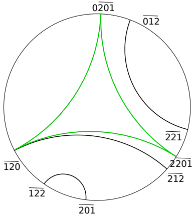
Lamination Data



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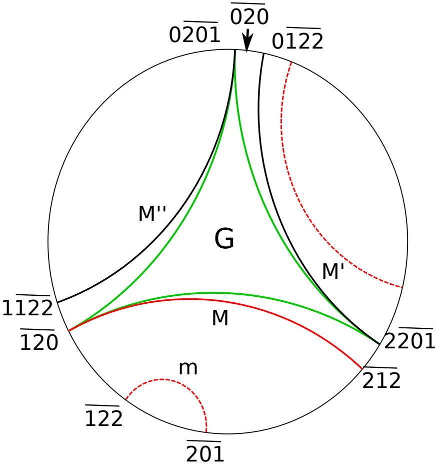
Lamination

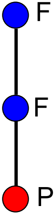
Julia Set

Pulling back longest leaf



Role Reversal: F-F-P

Bicolored Tree



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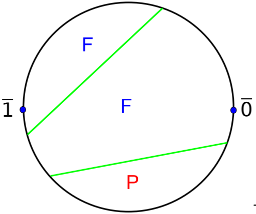

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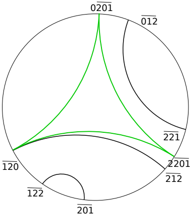

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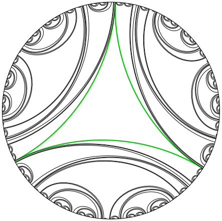

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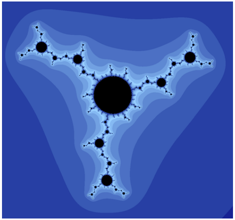
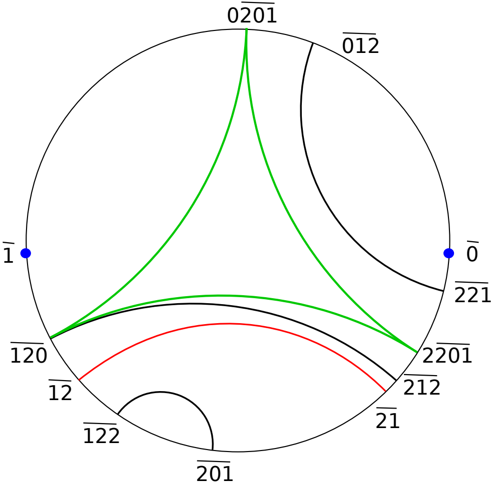


Lamination

Helicopter Julia Set

Periodic Forcing



A major goal is to understand periodic forcing for degree $d \geq 3$.

Some Problems

1. What role is played by periodic forcing in determining the simplest lamination from given periodic data.
2. Does each initial lamination data set (periodic polygons and critical chords) correspond to some complex polynomial?
3. How many weakly bicolored trees are there for a given degree (number of vertices)?

Meta-Problems

1. Are laminations useful in understanding polynomial dynamics?
   - Wandering branch points exist for polynomial Julia sets of degree 3. [Blokh and Oversteegen]
   - There are two distinct kinds of branch points that first return without rotation for polynomial Julia sets of degree 3. [Barry and M.]
2. Are laminations applicable outside polynomial dynamics?
   - Julia sets of the exponential family $E_\lambda(z) = \lambda e^z$ can be described by laminations of the half-plane. [Hartley and M.]

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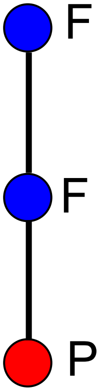
Preview of David George's talk

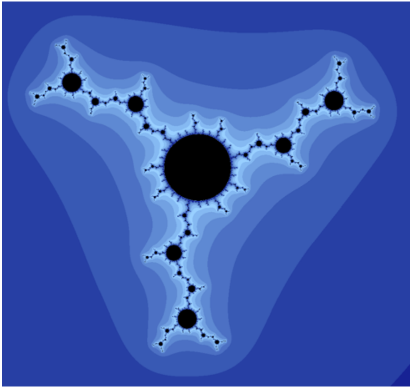
- Correspondence between generic critical portraits and bicolored trees.
- Non-generic critical portraits, all-critical polygons, and tricolored trees.
- Orbits under σ_d commute with rotation by a fixed point.
- The pullback step under σ_d commutes with rotation by a fixed point.
- Dynamical equivalence of pullback laminations.

THANKS!

Rotation by a Fixed Point: F-F-P $\longrightarrow$ P-F-F

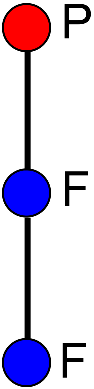


Bicolored Tree

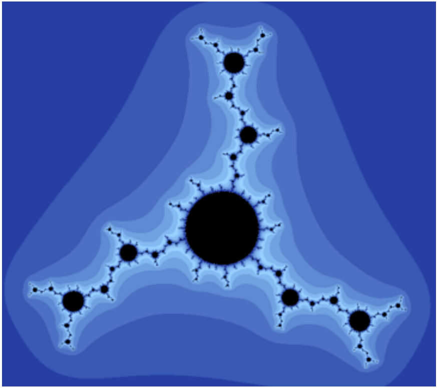


Helicopter Julia Set

Rotation by a Fixed Point: F-F-P $\longrightarrow$ P-F-F



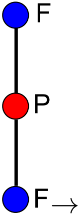
Rotated Bicolored Tree



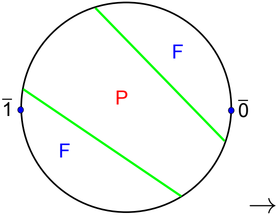
Rotated Helicopter

Role Reversal: F-P-F

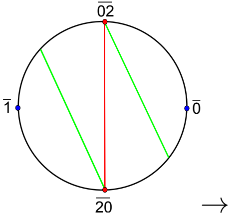
Bicolored Tree

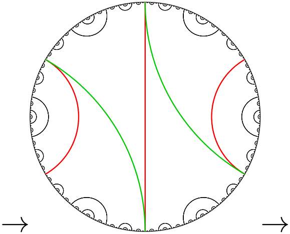


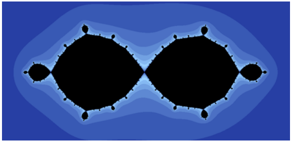
Critical Portrait



Lamination Data





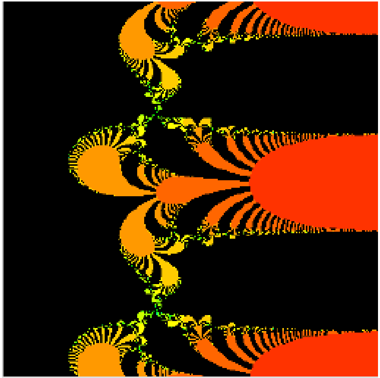


Lamination

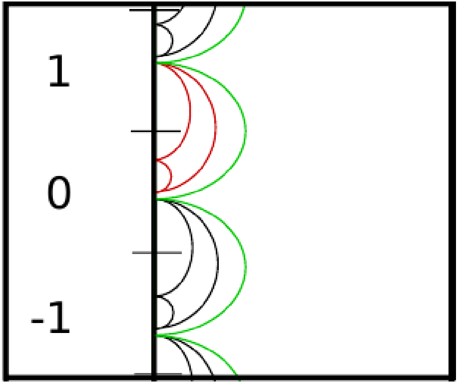
Julia Set

Exponential Laminations

Julia set of $E_\lambda(z) = \lambda e^z$, $\lambda = 3 + \pi$



Half-plane lamination



Laminations can be adapted to the Exponential family of functions using a half-plane model. Cf: Hartley